Quantum mechanics II, Chapter 4 : Reduced States and Decoherence

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Our systems do not always stay closed.

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Problem 1 : Bloch sphere for pure or mixed states of a two-level system

In the lecture you have started to talk about density matrices. This exercise serves as a first introduction to the topic, connecting to the already known concept of the Bloch sphere.

1. *Derivation of Bloch vector from generic pure state.* Any pure one-qubit quantum state can be written as ket
$$|\psi\rangle = \cos(\theta/2)|0\rangle + \sin(\theta/2)e^{i\phi}|1\rangle \qquad \theta \in [0,\pi),\ \phi \in [0,2\pi)$$
or as the density matrix,
$$|\psi\rangle\langle\psi| = \frac{1}{2}\left(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}\right)$$
Find an expression for $\mathbf{r}$ in terms of θ and ϕ. What does the vector $\mathbf{r}$ denote ?
2. *Derivation of Bloch vector from properties of density operators.* Define the set of density matrices with the following 3 conditions :
   - The density matrix is Hermitian : $\hat{\rho}^\dagger = \hat{\rho}$
   - It has trace 1 : $\mathrm{Tr}\hat{\rho} = 1$
   - It is positive or null : $\langle\Psi|\hat{\rho}|\Psi\rangle \geq 0, \quad \forall\Psi$

   Show that any density matrix $\hat{\rho}$ of the 2 level system can be written
$$\hat{\rho} = \frac{1}{2}(\mathbb{1} + \hat{\boldsymbol{\sigma}} \cdot \boldsymbol{r}), \tag{1}$$
where $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z)$. Argue that $\boldsymbol{r}$ is a real vector of 3D space and $|\boldsymbol{r}| \leq 1$.
3. Show that the state is pure iff $||\boldsymbol{r}|| = 1$. Explain why $\mathrm{Tr}[\rho^2]$ is a measure of the 'purity' of a quantum state.
4. Sketch on the Bloch sphere the states :
   a) $|0\rangle$
   b) $|+\rangle := \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$
   c) $|-\rangle := \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$
   d) $\frac{1}{\sqrt{3}}(|0\rangle - i\sqrt{2}\,|1\rangle)$.
   e) $\frac{1}{2}\mathbb{1}$
   f) $\frac{1}{3}|+\rangle\langle+| + \frac{2}{3}|-\rangle\langle-|$

5. Give a geometric argument to show that $\frac{1}{2}(|+\rangle\langle+| + |-\rangle\langle-|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$. (Is this surprising ?)

   Disclaimer : think about the meaning of this state. How would you represent $|0\rangle\langle 0|$ on the Bloch sphere ? And $|1\rangle\langle 1|$? Now if the space of qubits is a convex space, what is the point in the Bloch sphere that represents the combination of the previous one ?

   Now make the same reasoning for $|+\rangle\langle+|$, $|-\rangle\langle-|$.

Problem 2 : No signalling

Use your new-found understanding of reduced states to justify the no signalling principle (i.e. to argue why it is not possible to use entanglement to signal faster than light). Discuss with a partner.

Problem 3 : Density operator, partial trace, information and measurement

Alice and Bob share state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}} \tag{2}$$

1. What is the density matrix of the system $\hat{\rho}$ with 2 qubits ?
2. Verify that it is a pure state by calculating $\mathrm{Tr}(\hat{\rho}^2)$.

Note $\hat{\rho}_B = \mathrm{Tr}_A\, \hat{\rho}$ the density matrix obtained by partial trace on Alice's qubit. This matrix is an operator on the Hilbert space of the second qubit (Bob's), and reflects the information available to Bob.

3. Calculate $\hat{\rho}_B$ and link that result to the probability Bob has to get outcome 0 or 1 when he measures his qubit in the computational basis (We will write $\hat{O}$ the corresponding observable). Also verify that we have $\langle\hat{O}\rangle = \mathrm{Tr}[\hat{\rho}(\mathbb{1}\otimes\hat{O})] = \mathrm{Tr}(\hat{\rho}_B\hat{O})$.
4. Does the matrix ρ_B describe a pure state of the second qubit ? Justify by calculating $\mathrm{Tr}(\hat{\rho}_B^2)$. What about if the 2 qubit state $|\psi\rangle$ being separable in the form $|\psi_A\rangle\otimes|\psi_B\rangle$? We sometimes say that statistical mixtures of the state of a system is the fruit of entanglement of this system with its environment ; how can we interpret this in the light of the previous results ?

We admit that when the measurement of an observable $\hat{M}$ on the system gives the result m, then the density matrix ($\hat{\rho}$ before measurement) reads

$$\hat{\rho}' = \frac{\hat{P}_m\hat{\rho}\hat{P}_m^\dagger}{\mathrm{Tr}(\hat{P}_m^\dagger\hat{P}_m\hat{\rho})}, \tag{3}$$

where $\hat{P}_m$ is the projector on the subspace relative to m.

Disclaimer : If you don't remember why this is the case have a look at the measurement postulate and the post-measurement state `https://en.wikipedia.org/wiki/Measurement_in_quantum_mechanics#State_change_due_to_measurement`

5. What is the state with 2 qubits $|\psi'\rangle$ obtained when Alice measure her qubit in the computational basis and finds 0 ? Compare $|\psi'\rangle\langle\psi'|$ and $\hat{\rho}'$.
6. When Alice measures her qubit on state $|\psi\rangle$ and finds 0, what is the density matrix ? $\hat{\rho}_B'$? Comment.